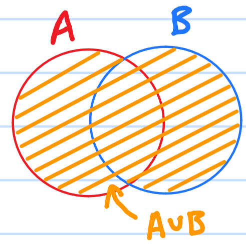
Set operations:

Suppose A and B are two sets

- The <u>union</u> of A and B is the set

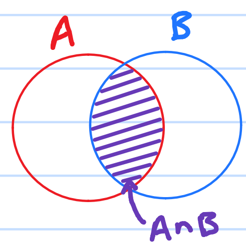
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



(Venn diagram)

- The <u>intersection</u> of A and B is the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



Ex: If $A = \{a, f, i\}$ and $B = \{a, b, f, g\}$ then

$A \cup B = \{a, b, f, g, i\}$ and $A \cap B = \{a, f\}$.

Properties of unions and intersections:

- Commutative properties:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- Associative properties:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive properties:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Unions & intersections of collections of sets:

Suppose that I is some <u>indexing set</u> and that, for each $i \in I$, A_i is a set.

Write $\{A_i\}_{i\in I} = \{A_i : i \in I\}$.

- The union of $\{A_i\}_{i \in I}$ is

$$\bigcup_{i\in I} A_i = \{x : \exists i \in I \text{ s.t. } x \in A_i\}$$

- The intersection of $\{A_i\}_{i\in I}$ is

$$\bigcap_{i \in I} A_i = \{x : \forall i \in I, x \in A_i\}$$

Ex: Let $I = \mathbb{N}$,

$A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,2,3\}$,

$\forall i \in I$, $A_i = \{0,1,\ldots,i\}$.

Then:

- $\bigcup_{i\in I} A_i = \{0,1,2,\ldots\} = \mathbb{Z}_{\geq 0}$

- $\bigcap_{i\in I} A_i = \{0,1\}$

Proofs of these facts:

- $\bigcup_{i\in I} A_i = \{0,1,2,\dots\} = \mathbb{Z}_{\geq 0}$

Pf: First, suppose that $n \in \mathbb{Z}_{\geq 0}$.

Then $n \in A_n$, so $n \in \bigcup_{i\in I} A_i$.

This shows that $\mathbb{Z}_{\geq 0} \subseteq \bigcup_{i\in I} A_i$.

On the other hand, suppose that $m \in \bigcup_{i\in I} A_i$.

Then $\exists i \in I$ s.t. $m \in A_i$.

Since $A_i \subseteq \mathbb{Z}_{\geq 0}$, we have $m \in \mathbb{Z}_{\geq 0}$.

Therefore $\bigcup_{i\in I} A_i \subseteq \mathbb{Z}_{\geq 0}$.

Conclusion: $\bigcup_{i\in I} A_i = \mathbb{Z}_{\geq 0}$. ▨

- $\bigcap_{i\in I} A_i = \{0,1\}$

Pf: If $n \in \{0,1\}$ then $n \in A_i$, $\forall i \in I$.

Therefore $n \in \bigcap_{i\in I} A_i$. So $\{0,1\} \subseteq \bigcap_{i\in I} A_i$.

If $m \in \bigcap_{i\in I} A_i$ then $m \in A_1 = \{0,1\}$.

So $\bigcap_{i\in I} A_i \subseteq \{0,1\}$.

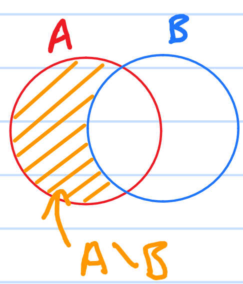
Conclusion: $\bigcap_{i\in I} A_i = \{0,1\}$. ▨

Disjoint sets:

- We say that A and B are <u>disjoint</u> if $A \cap B = \phi$.
- We say that a collection of sets $\{A_i\}_{i \in I}$ is <u>pairwise disjoint</u> if $A_i \cap A_j = \phi$ for all $i, j \in I$ with $i \neq j$.

- The <u>set difference of A minus B</u> is

$$A \setminus B = \{x \in A : x \notin B\}. \qquad (= A - B)$$



Ex: If $A = \{a, f, i\}$ and $B = \{a, b, f, g\}$ then

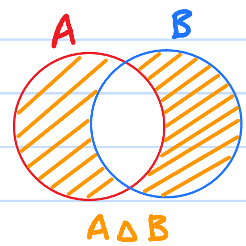
$A \setminus B = \{i\}$ and $B \setminus A = \{b, g\}$

Note: It is <u>not</u> true, for general sets A and B, that $A \setminus B = B \setminus A$.

(Ques: When exactly is this true?)

- The <u>symmetric difference</u> of A and B is

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



Ex: If $A = \{a, f, i\}$ and $B = \{a, b, f, g\}$ then

$$A \setminus B = \{i\} \quad \text{and} \quad B \setminus A = \{b, g\}.$$

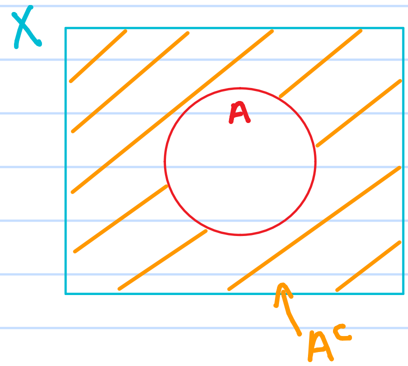
$$A \triangle B = \{i, b, g\}.$$

Note: For any sets A and B,

$$A \triangle B = B \triangle A.$$

• If $A \subseteq X$ then the <u>complement of A in X</u> is

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$



• A^c <u>depends on X</u>

• For any $A \subseteq X$,

$$(A^c)^c = A.$$

Exs: • $A = \mathbb{N}$, $X = \mathbb{Z}$,

$$A^c = \{\ldots -2, -1, 0\}$$

$\neq$

• $A = \mathbb{N}$, $X = \mathbb{Q}$,

$$A^c = \{x \in \mathbb{Q} : x \notin \mathbb{N}\}$$

- The <u>Cartesian product</u> of A and B, denoted $A \times B$, is the set of <u>ordered pairs</u> (a,b) with $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) : a \in A,\ b \in B\}.$$

Exs:

- $A = \{1, 2, x\}$, $B = \{1, 2, y\}$

  $A \times B = \{(1,1), (1,2), (1,y), (2,1), (2,2), (2,y), (x,1), (x,2), (x,y)\}.$

  Note: $(1,2) \neq (2,1)$ (order matters)

- $\mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R},\ y \in \mathbb{R}\} = \mathbb{R}^2$

  Geometric description: (x,y)-points in Cartesian plane.

More generally, if $A_1, \dots, A_n$ are sets then

$$A_1 \times \cdots \times A_n = \{ (a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n \}.$$

ordered n-tuple

Cartesian product of $A_1, \dots, A_n$.

Special case:

$A^n = \underbrace{A \times \cdots \times A}_{n\text{-times}}$ (ordered n-tuples of elements of A)

Important fact:

- If $A_1, \dots, A_n$ are finite then

$$|A_1 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|.$$

Warning: The symbol $\times$ is used in sometimes ambiguous ways. For example, later on we will also use the same symbol to define direct products of groups.

- The <u>power set</u> of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

$$\mathcal{P}(A) = \{B : B \subseteq A\}. \qquad (\text{also denoted } 2^A).$$

Exs:

- $A = \emptyset$ $\qquad (|A| = 0,\ |\mathcal{P}(A)| = 1)$

  $\mathcal{P}(A) = \{\emptyset\}$

- $A = \{\Delta\}$ $\qquad (|A| = 1,\ |\mathcal{P}(A)| = 2)$

  $\mathcal{P}(A) = \{\emptyset, \{\Delta\}\}$

- $A = \{1, \{2\}\}$ $\qquad (|A| = 2,\ |\mathcal{P}(A)| = 4)$

  $\mathcal{P}(A) = \{\emptyset, \{1\}, \{\{2\}\}, \{1, \{2\}\}\}.$

Thm: If $|A| = n < \infty$ then $|\mathcal{P}(A)| = 2^n$.

"Pf:" Every possible subset of A is uniquely determined by looking at each element of A and choosing whether or not to include it in the subset. There are 2 choices for each of the n elements, so the number of subsets is $\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n\text{-times}} = 2^n$. $\boxtimes$

(Note: this proof "hides" the fact that we are relying on the principle of mathematical induction)